UDC 531.31

ON THE STABILITY OF AN AUTONOMOUS HAMILTONIAN SYSTEM WITH TWO DEGREES OF FREEDOM IN THE CASE OF EQUAL FREQUENCIES

PMM Vol. 38, № 5, 1974, pp. 791-799

A. G. SOKOL'SKII
(Moscow)

(Received December 6, 1973)

The stability question is settled in a nonlinear formulation. The cases of simple and nonsimple elementary divisors of the characteristic matrix of a linear system have been examined. The real normal form of the Hamiltonian of the linear problem and the corresponding normalizing transformation have been found for the second case. In the first case Liapunov instability and stability, while in the second case instability and formal stability, have been proved as a function of the coefficients of the Hamilton function.

1. We consider an autonomous Hamiltonian system with two degrees of freedom. The coordinates q_1, q_2 and the momenta p_1, p_2 are chosen such that the point $q_1 = q_2 = p_1 = p_2 = 0$ is an equilibrium position of the differential equation system, while the Hamilton function is represented in the series form

$$H = H_2 + H_3 + H_4 + \ldots + H_m + \ldots \tag{1.1}$$

where H_m are mth-degree polynomials in the coordinates and momenta. If H_2 is a sign-definite function, then by Liapunov's theorem [1] the equilibrium position is stable. If H_2 is not a sign-definite function, but stability holds in the first approximation and the frequencies ω_1, ω_2 of the linear problem are not connected by resonance relations of up to fourth order, inclusive, then in the majority of cases the stability question is settled using the Arnol'd-Moser theorem [2, 3].

Suppose that integers n_1 and n_2 exist such that $0 < |n_1| + |n_2| \leqslant 4$ and $n_1\omega_1 + n_2\omega_2 = 0$ then the Arnol'd-Moser theorem is inapplicable and the stability problem requires a particular investigation. Stability under the resonances $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$ was investigated by Markeev [4]. The aim of the present paper is to obtain stability and instability conditions under the resonance $\omega_1 = \omega_2$, as well as to obtain an expression of these conditions in terms of the coefficients of forms H_2, H_3, H_4.

The first stage in solving the problem is the determination of the normal form of the linear system. By analogy with the case of nonmultiple frequencies we could assume that in the given case the normal form is a Jordan form. However, the differential equation system corresponding to it is not a canonical one. Let us examine in more detail a linear system with the Hamiltonian

$$\begin{aligned} H_2 = {}& {}^1/_2 a_{11}q_1^2 + a_{12}q_1q_2 + {}^1/_2 a_{22}q_2^2 + c_{11}q_1p_1 + c_{12}q_1p_2 + \\ & c_{21}q_2p_1 + c_{22}q_2p_2 + {}^1/_2 b_{11}p_1^2 + b_{12}p_1p_2 + {}^1/_2 b_{22}p_2^2 = {}^1/_2 \mathbf{q}'H\mathbf{q} \end{aligned} \tag{1.2}$$

The canonical equations of motion of such a system are written as

$$d\mathbf{q} / dt = JH\mathbf{q} \tag{1.3}$$

$$\mathbf{q} = \begin{Vmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{Vmatrix}, \quad H = \begin{Vmatrix} a_{11} & a_{12} & c_{11} & c_{12} \\ a_{12} & a_{22} & c_{21} & c_{22} \\ c_{11} & c_{21} & b_{11} & b_{12} \\ c_{12} & c_{22} & b_{12} & b_{22} \end{Vmatrix}, \quad J = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{Vmatrix}$$

By virtue of the multiplicity of the roots of the characteristic equation of the linear system (1.3), it can be written in the following form:

$$\lambda^4 + 2\omega^2\lambda^2 + \omega^4 = 0 \tag{1.4}$$

where $\lambda_1 = \lambda_2 = i\omega$ and $\lambda_3 = \lambda_4 = -i\omega$ are the roots of the characteristic equation.

Let $D_k(\lambda)$ be the greatest common divisor of all minors of the defining matrix $(JH - \lambda E)$ of order k [5]. In the problem being investigated $D_0 = D_1 = D_2 = 1$ always. In addition $D_4 = \lambda^4 + 2\omega^2\lambda^2 + \omega^4$. Two cases are possible, depending on the coefficients a_{ij}, b_{ij}, c_{ij}: (1) $D_3 = \lambda^2 + \omega^2$, (2) $D_3 = 1$. The invariant polynomials of matrix $(JH - \lambda E)$ for the first and second cases are, respectively: $i_1 = i_2 = \lambda^2 + \omega^2$, $i_3 = i_4 = 1$ and $i_1 = (\lambda^2 + \omega^2)^2$, $i_2 = i_3 = i_4 = 1$.

In the first case the defining matrix has simple elementary divisors and the normal form $i\omega\,(q_1p_1 + q_2p_2)$ has the real representation

$$H_2 = {}^1/_2\,(p_1^2 + \omega^2 q_1^2) - {}^1/_2\,(p_2^2 + \omega^2 q_2^2)$$

This case is investigated in Sect. 2. In the second case the elementary divisors are not simple. Then [6] there exists a linear complex canonical transformation P bringing the Hamiltonian (1.2) od system (1.3) to the form

$$H_2^* = i\omega\,(q_1^* p_1^* + q_2^* p_2^*) + q_2^* p_1^* \tag{1.5}$$

A constructive way for determining the matrix of transformation P is indicated in [6], but the question on the real form of the normalized Hamiltonian is not discussed. Let us find this form here.

The Hamiltonian (1.5) can be reduced to the form

$$H_2 = {}^1/_2\,(q_1^2 + q_2^2) + \omega\,(q_1p_2 - q_2p_1) \tag{1.6}$$

by a linear complex canonical transformation with the matrix

$$D = \begin{Vmatrix} b & ib & -a & -ia \\ a & ia & 0 & 0 \\ \dfrac{1}{2a} & -\dfrac{i}{2a} & 0 & 0 \\ -\dfrac{b}{2a^2} & \dfrac{ib}{2a^2} & \dfrac{1}{2a} & -\dfrac{i}{2a} \end{Vmatrix}$$

where a and b are any complex numbers. We now apply the transformation $N = PD$ to the original system with Hamiltonian (1.2) and we choose the numbers a and b such that the transformation will be real. This can always be done since the λ-matrices corresponding to Hamiltonians (1.2) and (1.6) have equal elementary divisors in the real number field [5]. The stability of the equilibrium position in the case of nonsimple elementary divisors is investigated in Sect. 3.

2. Let $\omega_1 = \omega_2$ and let the elementary divisors be simple. The Hamilton function (1.1) can be reduced by a linear real canonical transformation to the form

$$H = {}^1/_2(p_1^2 + \omega^2 q_1^2) - {}^1/_2(p_2^2 + \omega^2 q_2^2) + H_3 + H_4 + \ldots + H_m + \ldots \qquad (2.1)$$

$$H_m = \sum_{\nu=m} h_{\nu_1\nu_2\nu_3\nu_4} q_1^{\nu_1} q_2^{\nu_2} p_1^{\nu_3} p_2^{\nu_4}, \quad \nu = \nu_1 + \nu_2 + \nu_3 + \nu_4$$

We used the Birkhoff transformation [7] in complex coordinates to reduce the forms H_3 and H_4 in expansion (2.1) to normal form; in essence, the normalization carried out differs in no way from the analogous transformations in [4] for the resonances $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$. After all, having annulled form H_3 and simplified H_4, we can reduce the Hamiltonian (2.1) to the form (the notation for the variables remains as before)

$$\begin{aligned} H = {} & \frac{\omega}{2}[(q_1^2 + p_1^2) - (q_2^2 + p_2^2)] + \frac{c_{20}}{4}(q_1^2 + p_1^2)^2 + \\ & \frac{c_{11}}{4}(q_1^2 + p_1^2)(q_2^2 + p_2^2) + \frac{c_{02}}{4}(q_2^2 + p_2^2)^2 + \\ & \frac{k_{2002}}{2}[(q_1q_2 - p_1p_2)^2 - (q_1p_2 + q_2p_1)^2] + l_{2002}(q_1q_2 - p_1p_2) \times \\ & (q_1p_2 + q_2p_1) + \frac{1}{2}(q_1^2 + p_1^2)[k_{1120}(q_1p_2 + q_2p_1) + \\ & l_{1120}(q_1q_2 - p_1p_2)] + \frac{1}{2}(q_2^2 + p_2^2)[k_{1102}(q_1p_2 + q_2p_1) + \\ & l_{1102}(q_1q_2 - p_1p_2)] + H_5 + \ldots \end{aligned} \qquad (2.2)$$

$$c_{20} = -x_{2020} - \frac{3}{2}u_{1,1} + \frac{1}{2}u_{2,2} - \frac{1}{2}u_{4,4} - \frac{3\omega^2}{8}u_{7,7} + \frac{\omega^2}{24}u_{8,8}$$

$$c_{11} = x_{1111} + 2u_{1,6} + 2u_{3,3} - \frac{\omega^2}{6}u_{8,8} - 2u_{2,5} - 2u_{4,4} + \frac{\omega^2}{6}u_{9,9}$$

$$c_{02} = -x_{0202} + \frac{3}{2}u_{5,5} - \frac{1}{2}u_{6,6} + \frac{1}{2}u_{3,3} + \frac{3\omega^2}{8}u_{10,10} - \frac{\omega^2}{24}u_{9,9}$$

$$k_{2002} = x_{2002} - \frac{1}{2}u_{1,3} - u_{2,4} + \frac{\omega^2}{8}u_{7,9} + u_{3,6} + \frac{1}{2}u_{4,5} - \frac{\omega^2}{8}u_{8,10}$$

$$l_{2002} = y_{2002} - \frac{1}{2}v_{1,3} + v_{2,4} + \frac{\omega^2}{8}v_{7,9} - v_{3,6} + \frac{1}{2}v_{4,5} - \frac{\omega^2}{8}v_{8,10}$$

$$\begin{aligned} k_{1120} = {} & x_{1120} - \frac{1}{2}u_{2,1} + u_{1,4} + u_{3,2} - \frac{1}{2}u_{2,6} - \\ & \frac{1}{2}u_{6,4} - \frac{\omega^2}{4}u_{8,7} + \frac{\omega^2}{12}u_{9,8} \end{aligned}$$

$$l_{1120} = y_{1120} - \frac{1}{2}v_{2,1} + v_{1,4} + v_{3,2} - \frac{1}{2}v_{2,6} - \frac{1}{2}v_{6,4} - \frac{\omega^2}{4}v_{8,7} + \frac{\omega^2}{12}v_{9,8}$$

$$k_{1102} = x_{1102} + \frac{1}{2}u_{6,2} + u_{2,3} - 2u_{5,3} - u_{4,6} + \frac{1}{2}u_{6,5} + \frac{\omega^2}{4}u_{9,10} - \frac{\omega^2}{12}u_{8,9}$$

$$l_{1102} = y_{1102} + \frac{1}{2}v_{6,2} + v_{2,3} - 2v_{5,3} - v_{4,6} - \frac{1}{2}v_{6,5} + \frac{\omega^2}{4}v_{9,10} - \frac{\omega^2}{12}v_{8,9}$$

$$u_{i,j} = x_i x_j + y_i y_j, \ v_{i,j} = x_i y_j - x_j y_i \quad (i = 1, \ldots, 10)$$
$$x_{2020} = -\frac{1}{2}\left(3\omega^2 h_{0040} + h_{2020} + \frac{3}{\omega^2} h_{4000}\right)$$
$$x_{1111} = \omega^2 h_{0022} + h_{0220} + h_{2002} + \frac{1}{\omega^2} h_{2200}$$
$$x_{0202} = -\frac{1}{2}\left(3\omega^2 h_{0004} + h_{0202} + \frac{3}{\omega^2} h_{0400}\right)$$
$$x_{2002} = \frac{1}{4}\left(\omega^2 h_{0022} + h_{0220} + h_{1111} + h_{2002} + \frac{1}{\omega^2} h_{2200}\right)$$
$$y_{2002} = \frac{1}{4}\left(-\omega h_{0121} - \omega h_{1012} + \frac{1}{\omega} h_{1210} + \frac{1}{\omega} h_{2101}\right)$$
$$x_{1120} = \frac{1}{4}\left(3\omega h_{0130} + \omega h_{1021} + \frac{1}{\omega} h_{2110} + \frac{3}{\omega} h_{3001}\right)$$
$$y_{1120} = \frac{1}{4}\left(-3\omega^2 h_{0031} + h_{1120} - h_{2011} + \frac{3}{\omega^2} h_{3100}\right)$$
$$x_{1102} = \frac{1}{4}\left(3\omega h_{1003} + \omega h_{0112} + \frac{1}{\omega} h_{1201} + \frac{3}{\omega} h_{0310}\right)$$
$$y_{1102} = \frac{1}{4}\left(-3\omega^2 h_{0013} + h_{0211} - h_{1102} + \frac{3}{\omega^2} h_{1300}\right)$$

$$x_1 = -\frac{1}{2} h_{1020} - \frac{3}{2\omega^2} h_{3000}, \qquad y_1 = \frac{3\omega}{2} h_{0030} + \frac{1}{2\omega} h_{2010}$$
$$x_2 = -\omega h_{0021} - \frac{1}{\omega} h_{2001}, \qquad y_2 = h_{0120} + \frac{1}{\omega^2} h_{2100}$$
$$x_3 = -\frac{1}{2} h_{0111} - \frac{1}{2} h_{1002} + \frac{1}{2\omega^2} h_{1200}, \qquad y_3 = -\frac{\omega}{2} h_{0012} + \frac{1}{2\omega} h_{0211} + \frac{1}{2\omega} h_{1101}$$
$$x_4 = -\frac{\omega}{2} h_{0021} + \frac{1}{2\omega} h_{1110} + \frac{1}{2\omega} h_{2001}, \qquad y_4 = -\frac{1}{2} h_{0120} - \frac{1}{2} h_{1011} + \frac{1}{2\omega^3} h_{2100}$$
$$x_5 = \frac{3\omega}{2} h_{0003} + \frac{1}{2\omega} h_{0201}, \qquad y_5 = -\frac{1}{2} h_{0102} - \frac{3}{2\omega^2} h_{0300}$$
$$x_6 = h_{1002} + \frac{1}{\omega^2} h_{1200}, \qquad y_6 = -\omega h_{0012} - \frac{1}{\omega} h_{0210}$$
$$x_7 = h_{0030} - \frac{1}{\omega^2} h_{2010}, \qquad y_7 = \frac{1}{\omega} h_{1020} - \frac{1}{\omega^3} h_{3000}$$
$$x_8 = \frac{1}{\omega} h_{0120} - \frac{1}{\omega} h_{1011} - \frac{1}{\omega^3} h_{2100}, \qquad y_8 = h_{0021} + \frac{1}{\omega^2} h_{1110} - \frac{1}{\omega^2} h_{2001}$$
$$x_9 = -h_{0012} + \frac{1}{\omega^2} h_{0210} - \frac{1}{\omega^2} h_{1101}, \qquad y_9 = \frac{1}{\omega} h_{0111} - \frac{1}{\omega} h_{1002} + \frac{1}{\omega^3} h_{1200}$$
$$x_{10} = -\frac{1}{\omega} h_{0102} + \frac{1}{\omega^3} h_{0300}, \qquad y_{10} = -h_{0003} + \frac{1}{\omega^2} h_{0201}$$

In "polar" coordinates $r_1, r_2, \varphi_1, \varphi_2$ defined by the formulas

$$q_i = \sqrt{2r_i} \sin \varphi_i, \quad p_i = \sqrt{2r_i} \cos \varphi_i$$

Hamiltonian (2.2) takes the form

$$H = \omega (r_1 - r_2) + c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2 +$$
$$2 r_1 r_2 [k_{2002} \cos 2(\varphi_1 + \varphi_2) - l_{2002} \sin 2(\varphi_1 + \varphi_2)] +$$

$$2r_1\sqrt{r_1r_2}\,[k_{1120}\sin(\varphi_1+\varphi_2)-l_{1120}\cos(\varphi_1+\varphi_2)]+$$
$$2r_2\sqrt{r_1r_2}\,[k_{1102}\sin(\varphi_1+\varphi_2)+l_{1102}\cos(\varphi_1+\varphi_2)]+O(r_i^{5/2})$$

We denote $A=2\sqrt{k_{2002}^2+l_{2002}^2}$, $B=2\sqrt{k_{1120}^2+l_{1120}^2}$, $C=2\sqrt{k_{1102}^2+l_{1102}^2}$ and we suppose that $A\neq 0$, $B\neq 0$, $C\neq 0$. Then, defining the angles θ_1, θ_2, θ_3 by the relations

$$\sin 2\theta_1=\frac{2k_{2002}}{A},\quad \sin\theta_2=-\frac{2l_{1120}}{B},\quad \sin\theta_3=\frac{2l_{1102}}{C}$$
$$\cos 2\theta_1=-\frac{2l_{2002}}{A},\quad \cos\theta_2=\frac{2k_{1120}}{B},\quad \cos\theta_3=\frac{2k_{1102}}{C}$$

we obtain

$$H=\omega(r_1-r_2)+c_{20}r_1^2+c_{11}r_1r_2+c_{02}r_2^2+ \tag{2.3}$$
$$Ar_1r_2\sin 2(\varphi_1+\varphi_2+\theta_1)+Br_1\sqrt{r_1r_2}\sin(\varphi_1+\varphi_2+\theta_2)+$$
$$Cr_2\sqrt{r_1r_2}\sin(\varphi_1+\varphi_2+\theta_3)+O(r_i^{5/2})$$

Using the integral $H=h=\text{const}$ we decrease the system's order by two units [8]; the new Hamiltonian is 2π-periodic in the new independent variable. Since the motion is being examined in a sufficiently small neighborhood of the origin, we can assume that r_1, $r_2\sim\varepsilon$, where $0<\varepsilon\ll 1$. Let the initial conditions be such that $h\sim\varepsilon^{5/2}$. Then, by solving Eq. (2.3) relative to r_2, we obtain

$$r_2=-K_0(r_1,\varphi_1,\varphi_2)-K_1(r_1,\varphi_1,\varphi_2,h),\ K_1=O(r_1^{5/2})$$
$$K_0=-r_1-\frac{1}{\omega}r_1^2[(c_{20}+c_{11}+c_{02})+A\sin 2(\varphi_1+\varphi_2+\theta_1)+$$
$$B\sin(\varphi_1+\varphi_2+\theta_2)+C\sin(\varphi_1+\varphi_2+\theta_3)]$$

Here K_1 is a function 2π-periodic in φ_1 and in the new independent variable φ_2. If instead of φ_1 we introduce the new angle $\varphi=\varphi_1+\varphi_2+\theta_1$, and instead of r_1 the new momentum r, then the Hamiltonian

$$K=r^2(a+b\sin 2\varphi+c\sin\varphi+d\cos\varphi)+K^*(r,\varphi,\varphi_2,h),\quad K^*=O(r^{5/2}) \tag{2.4}$$

will correspond to the resulting system with one degree of freedom. Here K^* is a function 2π-periodic in φ and φ_2, and

$$a=-\frac{1}{\omega}(c_{20}+c_{11}+c_{02}),\qquad c=-\frac{1}{\omega}[B\cos(\theta_2-\theta_1)+C\cos(\theta_3-\theta_1)]$$
$$b=-\frac{1}{\omega}A,\qquad d=-\frac{1}{\omega}[B\sin(\theta_2-\theta_1)+C\sin(\theta_3-\theta_1)]$$

From the equations of motion with Hamiltonian (2.3) it follows that for sufficiently small r_1 and r_2 the angular variable φ_2 is a monotonic function of time and, consequently, φ_2 can play the role of time in the stability problem. Thus, as in [4, 8], the stability investigation of a system with two degrees of freedom has been successfully reduced to the investigation of a system with one degree of freedom.

Theorem 2.1. If the function $\Phi(\varphi)=a+b\sin 2\varphi+C\sin\varphi+d\cos\varphi$ does not vanish for any φ, then the equilibrium position under investigation is Liapunov-stable. If there exists φ^* $(0\leqslant\varphi^*\leqslant 2\pi)$ such that $\Phi(\varphi^*)=0$, while $\Phi'(\varphi^*)\neq 0$, then the equilibrium position $q_i=p_i=0$ is unstable.

Note 2.1. If there exists φ^* such that $\Phi(\varphi^*)=\Phi'(\varphi^*)=0$, then the stability question is resolved by the higher-order terms in the expansion of the problem's Hamilton function.

Let us first prove the assumption on instability. We note that from the periodicity of function Φ and from the fact that $\Phi'(\varphi^*) = 0$, it follows that if the equation $\Phi(\varphi) = 0$ has roots, then there are at least two of them. We denote the two roots closest to each other by φ^* and φ^{**} and we let the root φ^* be such that $\Phi(\varphi^*) = 0$, while $\Phi'(\varphi^*) < 0$. We then prove instability with respect to variable r by means of Chetaev's theorem [9] and of the results of [10]. As the Chetaev function we take the function

$$V = r^2 \sin \Psi, \qquad \Psi = \pi / 2\delta (\varphi - \varphi^* + \delta)$$

where we choose the sufficiently small number δ such that there are no other roots of the function $\Phi(\varphi)$ in the neighborhood $\varphi^* - \delta < \varphi < \varphi^* + \delta$, while the sign of $\Phi'(\varphi)$ is preserved in this neighborhood. As the region $V > 0$ we take the region $(r > 0, \varphi^* - \delta < \varphi < \varphi^* + \delta)$. By virtue of the equations of motion the derivative of function V with respect to the independent variable φ_2 is

$$\frac{dV}{d\varphi_2} = 2r^3 \{\Phi(\varphi) \frac{\pi}{2\delta} \cos \Psi - \Phi'(\varphi) \sin \Psi\} + O(r^{7/2})$$

and this function is positive definite since $\Phi'(\varphi) < 0$ and $\sin \Psi > 0$ in the region $V > 0$, while for $\varphi^* - \delta < \varphi < \varphi^*$ the function $\Phi(\varphi) > 0$ and $\cos \Psi > 0$, and for $\varphi^* < \varphi < \varphi^* + \delta$ the function $\Phi(\varphi) < 0$ and also $\cos \Psi < 0$; here the expression within the braces does not vanish either in region $V > 0$ or on its boundary. Thus we have proved the assertion on instability when roots of the equation $\Phi(\varphi) = 0$ exist.

To prove stability we pass to the variables the action J and the angle W [11]. Then the generating function of the canonical trasformation $r, \varphi \to J, W$ is

$$S(J, \varphi) = \frac{2\pi J}{M} \int_0^{\varphi} \frac{d\varphi}{\sqrt{\Phi(\varphi)}}, \qquad M = \int_0^{2\pi} \frac{d\varphi}{\sqrt{\Phi(\varphi)}}$$

we note that M exists when the theorem's hypotheses are fulfilled. In the new variables the Hamiltonian (2.4) is expressed as follows:

$$K(J, W) = \frac{4\pi^2}{M^2} J^2 + K^*(J, W, \varphi_2, h), \quad K^* = O(J^{5/2})$$

where K^* is a 2π-periodic function in the variables W and φ_2, and is analytic in all the variables in the region

$$0 < c_1 \leqslant J \leqslant c_2, \quad |h| < c_3, \quad |\operatorname{Im} W, \varphi_2| < c_4$$

where c_j are real numbers. Since $\partial^2 (K - K^*) / \partial J^2 \neq 0$, hence, according to [2], follows the stability of the equilibrium position $q_i = p_i = 0$.

3. The difficulty of investigating stability in the case of nonsimple elementary divisors is, mathematically, that even in the first approximation the variables corresponding to the different degrees of freedom are not separable. Therefore, we cannot succeed in reducing the stability investigation to the study of the system with one degree of freedom as in the simpler case of linear elementary divisors of the defining matrix. It is also very interesting that in contrast to the preceding case and to all the investigated cases of the stability of a system with two degrees of freedom, the linear problem is unstable because of the presence of terms of the form $t \sin \omega t$ in the general solution. However, accounting for the nonlinear terms in the equations of motion can lead both to the stability as well as the instability of the full system [12].

Let the Hamilton function of the problem be represented in the form

$$H = \frac{1}{2}(q_1^2 + q_2^2) + \omega(q_1p_2 - q_2p_1) + \sum_{\nu=3}^{\infty} h_{\nu_1\nu_2\nu_3\nu_4} q_1^{\nu_1} q_2^{\nu_2} p_1^{\nu_3} p_2^{\nu_4} \qquad (3.1)$$

The form H_3 in (3.1) again can be annulled completely and the form H_4 simplified by applying the Birkhoff transformation. After some further calculations, more cumbersome than in the first case, the Hamilton function (3.1) can be reduced to

$$\begin{aligned}
H &= {}^1/_2\,(q_1^2 + q_2^2) + \omega\,(q_1p_2 - q_2p_1) + \\
&\quad (p_1^2 + p_2^2)\,[A\,(p_1^2 + p_2^2) + B\,(q_1p_2 - q_2p_1) + \\
&\quad C\,(q_1^2 + q_2^2)] + H_5 + \ldots
\end{aligned} \qquad (3.2)$$

$$A = {}^1/_4 k_{2002}, \quad B = -{}^1/_4\,(l_{2011} + l_{1102})$$

$$C = -{}^1/_4\,(2c_{20} + c_{11} + 2c_{02})$$

$$k_{2002} = x_{2002} + 3\,(u_{9,10} - u_{10,9}) + 2\,(u_{2,3} - u_{3,2}) + u_{3,6} - u_{6,3}$$

$$\begin{aligned}
l_{2011} &= y_{2011} - 6v_{10,8} - 4v_{3,1} + 2v_{6,2} + 2v_{9,9} + v_{2,2} - v_{5,3} \\
l_{1102} &= y_{2011} - 6v_{8,10} - 4v_{1,3} + 2v_{2,6} + 2v_{9,9} + v_{2,2} - v_{3,5} \\
c_{20} &= x_{2020} - 9u_{10,7} + 4u_{8,1} + u_{1,6} + u_{9,8} - u_{5,2} - u_{4,3} \\
c_{11} &= x_{1111} + 4\,(u_{9,8} - u_{8,9} + u_{4,3} - u_{3,4}) + 2\,(u_{6,5} - u_{5,6} + u_{2,1} - u_{1,2}) \\
c_{02} &= x_{2020} + 9u_{7,10} - 4u_{1,6} - u_{6,1} - u_{8,9} + u_{2,5} + u_{3,4} \\
x_{2002} &= {}^1/_2\,(3h_{0040} + h_{0022} + 3h_{0004}) \\
y_{2011} &= {}^1/_4\,(-h_{1021} - 3h_{1003} + 3h_{0130} + h_{0112}) \\
x_{2020} &= {}^1/_4\,(h_{2020} - h_{2002} + h_{1111} - h_{0220} + h_{0202}) \\
x_{1111} &= -\,(h_{2020} + h_{2002} + h_{0220} + h_{0202})
\end{aligned}$$

$$u_{i,j} = e_ix_j + f_iy_j, \quad v_{i,j} = e_iy_j - f_ix_j \qquad (i, j = 1, \ldots, 10),\ \Omega = \omega^{-1}$$

$$\begin{aligned}
e_1 &= \Omega y_1 + \Omega^2 x_2 - 2\Omega^3 y_3, & f_1 &= -\Omega x_1 + \Omega^2 y_2 + 2\Omega^3 x_3 \\
e_2 &= \Omega y_2 + 2\Omega^2 x_3, & f_2 &= -\Omega x_2 + 2\Omega^2 y_3 \\
e_3 &= \Omega y_3, & f_3 &= -\Omega x_3 \\
e_4 &= \Omega y_4 - \Omega^2 (x_1 - x_5) + & f_4 &= -\Omega x_4 - \Omega^2 (y_1 - y_5) - \\
&\quad 2\Omega^3 (y_2 - y_6) + 6\Omega^4 x_3, & &\quad 2\Omega^3 (x_2 - x_6) + 6\Omega^4 y_3 \\
e_5 &= \Omega y_5 - \Omega^2 (x_2 - 2x_6) + 4\Omega^3 y_3, & f_5 &= -\Omega x_5 - \Omega^2 (y_2 - 2y_6) - 4\Omega^3 x_3 \\
e_6 &= \Omega y_6 - \Omega^2 x_3, & f_6 &= -\Omega x_6 - \Omega^2 y_3 \\
e_7 &= {}^1/_3\Omega y_7 + {}^1/_9\Omega^2 x_8 - {}^2/_{27}\Omega^3 y_9 - & f_7 &= -{}^1/_3\Omega x_7 + {}^1/_9\Omega^2 y_8 + {}^2/_{27}\Omega^3 x_9 - \\
&\quad {}^2/_{27}\Omega^4 x_{10}, & &\quad {}^2/_{27}\Omega^4 y_{10} \\
e_8 &= {}^1/_3\Omega y_8 + {}^2/_9\,\Omega^2 x_9 - {}^2/_9\Omega^3 y_{10}, & f_8 &= -{}^1/_3\Omega x_8 + {}^2/_9\Omega^2 y_9 + {}^2/_9\Omega^3 x_{10} \\
e_9 &= {}^1/_3\Omega y_9 + {}^1/_3\Omega^2 x_{10}, & f_9 &= -{}^1/_3\Omega x_9 + {}^1/_3\Omega^2 y_{10} \\
e_{10} &= {}^1/_3\Omega y_{10}, & f_{10} &= -{}^1/_3\Omega x_{10}
\end{aligned}$$

$$x_i = x_i^* \,/\, 2\sqrt{2}, \quad y_i = y_i^* \,/\, 2\sqrt{2} \qquad (i = 1, \ldots, 10)$$

$$x_1^* = -h_{2010} - h_{1101} + h_{0210}, \qquad y_1^* = h_{2001} - h_{1110} - h_{0201}$$
$$x_2^* = -2h_{1020} - 2h_{1002}, \qquad y_2^* = -2h_{0120} - 2h_{0102}$$
$$x_3^* = -3h_{0030} - h_{0012}, \qquad y_3^* = -h_{0021} - 3h_{0003}$$
$$x_4^* = 3h_{3000} + h_{1200}, \qquad y_4^* = h_{2100} + 3h_{0300}$$
$$x_5^* = 2h_{2010} + 2h_{0210}, \qquad y_5^* = 2h_{2001} + 2h_{0201}$$
$$x_6^* = h_{1020} - h_{1002} + h_{0111}, \qquad y_6^* = h_{1011} - h_{0120} + h_{0102}$$
$$x_7^* = h_{3000} - h_{1200}, \qquad y_7^* = h_{2100} - h_{0300}$$
$$x_8^* = h_{2010} - h_{1101} - h_{0210}, \qquad y_8^* = h_{2001} + h_{1110} - h_{0201}$$
$$x_9^* = h_{1020} - h_{1002} - h_{0111}, \qquad y_9^* = h_{1011} + h_{0120} - h_{0102}$$
$$x_{10}^* = h_{0030} - h_{0012}, \qquad y_{10}^* = h_{0021} - h_{0003}$$

Theorem 3.1. If $A > 0$, the equilibrium position is formally stable. If $A < 0$, the equilibrium position is Liapunov-unstable.

To prove the formal stability we note that by means of an infinite number of stages of the Birkhoff transformation (possibly divergent) we can reduce the Hamilton function (3.2) to

$$H = \frac{1}{2}(q_1^2 + q_2^2) + \omega(q_1p_2 - q_2p_1) + (p_1^2 + p_2^2)[A(p_1^2 + p_2^2) + B(q_1p_2 - q_2p_1) + C(q_1^2 + q_2^2)] + \sum_{\alpha_1+\alpha_2+\alpha_3=3}^{\infty} h_{\alpha_1\alpha_2\alpha_3}(q_1^2 + q_2^2)^{\alpha_1}(p_1^2 + p_2^2)^{\alpha_2}(q_1p_2 - q_2p_1)^{\alpha_3} \tag{3.3}$$

A canonical system with Hamiltonian (3.3) has two formal integrals $H = \text{const}$ and $(q_1p_2 - q_2p_1) = \text{const}$. Consequently, the expression $G \equiv H - \omega(q_1p_2 - q_2p_1)$ also will be a formal integral of the system with Hamiltonian (3.3). But since the function

$$G_2 + G_4 = {}^1/_2(q_1^2 + q_2^2) + (p_1^2 + p_2^2)[A(p_1^2 + p_2^2) + B(q_1p_2 - q_2p_1) + C(q_1^2 + q_2^2)]$$

in the expansion

$$G = G_2 + G_4 + G_6 + \ldots + G_{2m} + \ldots$$

is a positive-definite function of its variables q_1, q_2, p_1, p_2 when $A > 0$, hence, according to definition [13], follows the formal stability of the equilibrium position.

To prove the instability we make use of Liapunov's instability theorem [1]. As the Liapunov function we take the sign-variable function

$$V = q_1p_1 + q_2p_2$$

The derivative of function V, taken relative to the equations of motion with Hamiltonian (3.2)

$$dV/dt = -(q_1^2 + q_2^2) + 4A(p_1^2 + p_2^2)^2 + 2B(q_1p_2 - q_2p_1) \times (p_1^2 + p_2^2) + O((q_1^2 + q_2^2 + p_1^2 + p_2^2)^{5/2})$$

is a negative-definite function of its variables when $A < 0$. The function V satisfies all the hypotheses of Liapunov's instability theorem. Thus, Theorem 3.1 has been proved completely.

In conclusion the author thanks A.P. Markeev for valuable advice and for discussing the results obtained.

REFERENCES

1. Liapunov, A. M., General Problem of the Stability of Motion. Collected Works, Vol. 2. Moscow-Leningrad, Izd. Akad. Nauk SSSR, 1956.
2. Arnol'd, V. I., Small denominators and the stability problems for motions in classical and celestial mechanics. Uspekhi Matem. Nauk, Vol. 18, № 6, 1963.
3. Moser, J., Lectures on Hamiltonian Systems. Providence, New Jersey, Mem. Amer. Math. Soc., № 81, 1968.
4. Markeev, A. P., Stability of a canonical system with two degrees of freedom in the presence of resonance. PMM Vol. 32, № 4, 1968.
5. Gantmakher, F. R., Matrix Theory. Moscow, "Nauka", 1966.
6. Roels, J. and Louterman, G., Normalisation des systèmes linéaires canoniques et application au problème restreint des trois corps. Celest. Mech., Vol. 3, № 1, 1970.
7. Birkhoff, G. D., DynamicalSystems. Providence, Amer. Math. Soc., 1966.
8. Markeev, A. P., On the problem of stability of equilibrium positions of Hamiltonian systems. PMM Vol. 34, № 6, 1970.
9. Chetaev, N. G., The Stability of Motion (English translation). Pergamon Press, Book № 09505, 1961.
10. Rumiantsev, V. V., On asymptotic stability and instability of motion with respect to a part of the variables. PMM Vol. 35, № 1, 1971.
11. Landau, L. D. and Lifshits, E. M., Mechanics. Moscow, "Nauka", 1973. (See also English translation: Addison-Wesley Publishing Co., 1960).
12. Liapunov, A. M., Investigation of one of the singular cases of the problem of stability of motion. Collected Works, Vol. 2, Moscow, Izd. Akad. Nauk SSSR, 1956.
13. Moser, J., New aspects in the theory of stability of Hamiltonian systems. Commun. Pure Appl. Math., Vol. 11, № 1, 1958.

Translated by N. H. C.

UDC 531.31:534

ON SYNCHRONIZATION OF DYNAMIC SYSTEMS

PMM Vol. 38, № 5, 1974, pp. 800-809

A. S. GURTOVNIK and Iu. I. NEIMARK
(Gor'kii)

(Received December 17, 1973)

We introduce the concepts of the degree and the order of synchronism on the basis of a mathematical model of the emergence of synchronization in the form of an asymptotically stable integral torus in the phase plane. We investigate the existence conditions for synchronisms in a dynamic system described by differential equations with rapidly rotating phases. As an application we examine synchronisms in a system of quasi-Hamiltonian objects. In recent years the phenomena of synchronization and resonance in dynamic systems have been subjected to intensive study, in particular, in connection with the question of the synchronization of satellites [1, 2] and of mechanical vibrators [3]. On the mathematical side the appearance of synchronization is closely connected with the theory of differential equations with rapidly rotating phases. Here in the first place we must